Some Approximation Concepts for Structural Synthesis

L. A. SCHMIT JR.* AND B. FARSHI†
University of California, Los Angeles, Calif.

An efficient automated minimum weight design procedure is presented which is applicable to sizing structural systems that can be idealized by truss, shear panel, and constant strain triangles. Static stress and displacement constraints under alternative loading conditions are considered. The optimization algorithm is an adaptation of the method of inscribed hyperspheres and high efficiency is achieved by using several approximation concepts including temporary deletion of noncritical constraints, design variable linking, and Taylor series expansions for response variables in terms of design variables. Optimum designs for several planar and space truss example problems are presented. The results reported support the contention that the innovative use of approximation concepts in structural synthesis can produce significant improvements in efficiency.

Nomenclature

B = total number of generalized design variables
$C_i = \rho_i l_i$ = constant equal to product of weight density (ρ_i) and length (l_i) of ith truss element
$\mathbf{D}$ = vector of design variables
D_i = ith scalar design variable
$G_q(\mathbf{D})$ = qth inequality constraint as a function of the design variable vector $\mathbf{D}$
$H_q(\boldsymbol{\alpha})$ = qth inequality constraint as a function of the generalized design variables
$\tilde{H}_q^{(p)}(\boldsymbol{\alpha})$ = linear approximation of the qth constraint function about $\boldsymbol{\alpha}_p$
I = total number of design variables
K = total number of load conditions
$M(\mathbf{D})$ = the objective function in terms of the design variables $\mathbf{D}$
$\tilde{M}(\boldsymbol{\beta})$ = the objective function in terms of the reciprocal variables $\boldsymbol{\beta}$
$\mathbf{P}_k$ = kth applied load vector
Q = total number of inequality constraints
$Q_R^{(p)}$ = reduced set of constraints retained in the truncated posture tables based on analysis of design $\boldsymbol{\alpha}_p$
$R_q(\boldsymbol{\alpha})$ = qth response ratio defined by Eq. (11)
$\mathbf{S}_p$ = vector locating the center of the largest inscribed hypersphere relative to the current trial design $\boldsymbol{\alpha}_p$
T = an $I \times B$ matrix in which each column corresponds to a basis vector $\mathbf{T}_b$
$\mathbf{T}_b$ = the bth basis vector
T_{ib} = an element of the matrix T
$W(\boldsymbol{\alpha})$ = the objective function in terms of the generalized design variables $\boldsymbol{\alpha}$
$\tilde{W}^{(p)}(\boldsymbol{\alpha})$ = linear approximation of the objective function about $\boldsymbol{\alpha}_p$
$Y_q(\boldsymbol{\alpha})$ = the qth response quantity of interest as a function of the generalized design variables
l_q = length of normal to the qth constraint hyperplane passing through the center of the largest inscribed hypersphere
l_w = length of a normal to the current linearized objective function passing through the center of the largest inscribed hypersphere
r_p = radius of the pth inscribed hypersphere
$\mathbf{u}_k$ = displacement response under kth load condition
$u_{jk}(\boldsymbol{\alpha})$ = jth displacement degree of freedom under the kth load condition
$\boldsymbol{\alpha}$ = the vector of generalized design variables
α_b = the bth scalar generalized design variable
$\boldsymbol{\beta}$ = vector of reciprocal design variables
β_i = ith scalar reciprocal design variable
$\kappa(\mathbf{D})$ = system stiffness matrix in terms of design variables $\mathbf{D}$
$\sigma_{ik}(\boldsymbol{\alpha})$ = stress in the ith member under the kth load condition

Superscripts

p = refers to pth stage
U = refers to upper limit
L = refers to lower limit

Subscripts

b = refers to bth generalized design variable
i = refers to ith design variable
j = refers to jth independent displacement degree of freedom
k = refers to kth load condition
p = refers to pth trial design in sequence
q = refers to qth inequality constraint

I. Introduction

IT is today widely recognized that an important class of structural design optimization problems can be formulated and solved using mathematical programing methods. A comprehensive review of mathematical programing techniques for structural design optimization is given in Ref. 1. Implementation of the structural synthesis concept has enjoyed considerable success and acceptance at the component level.[2-5] On the other hand, large scale structural optimization capabilities developed by combining finite element structural analysis with mathematical programing algorithms have required long running times to optimize problems that are only of modest proportions.[6,7] This situation had led some investigators to abandon the generality of the mathematical programing approach and direct renewed effort toward implementing methods based on fully stressed design concepts and discretized optimality criteria that are well suited to achieving high efficiency in appropriate specialized situations.[8-11] A widely held current viewpoint is that while mathematical programing methods are at present well suited to detailed component optimization they are not practical for dealing with large structural systems. This assessment of the state-of-the-art is well illustrated by the mixed optimization method for automated design of fuselage structures reported in Ref. 5. In this work a fully stressed design method is used to obtain a gross over-all distribution of material while the detailed design of rings and stiffened panels is carried out using mathematical programing techniques.

Current research and development activity in automated interdisciplinary aerospace vehicle design[12-15] provides addi-

Presented as Paper 73-341 at the AIAA/ASME/SAE 14th Structures, Structural Dynamics, and Materials Conference, Williamsburg, Va., March 20–22, 1973; submitted April 2, 1973; revision received October 19, 1973. This research was supported by NASA Research Grant NGR 05-007-337.

Index category: Structural Design, Optimal.

* Professor of Engineering and Applied Science. Associate Fellow AIAA.

† Research Assistant.

tional impetus to the quest for efficient structural optimization capabilities. The mathematical programing approach to structural optimization is quite general and the orderly logic remains philosophically attractive. No assumptions are made at the outset as to how many and which design constraints will become critical at the optimum design. The importance of this characteristic would seem to be reinforced when structural optimization is conducted within the larger context of automated interdisciplinary preliminary design procedures. In this connection it is noted that the structural synthesis concept is in principle open ended since new constraints can be added to the formulation as the need arises.

II. Approximation Concepts

While it is a time honored practice in structural engineering to employ a gradation of approximation levels in both the analysis and the design problem formulation relatively little attention has been given to approximation concepts in automated optimum design.

The purpose of this paper is to report some results of an ongoing research effort aimed at improving structural synthesis efficiency using approximation concepts. Previously reported applications of the mathematical programing approach to structural optimization have generally suffered from one or more of the following excesses: a) too many independent design variables were considered; b) too many behavior constraints were considered throughout the design optimization procedure; and c) too many detailed structural analyses were executed.

It is to be understood that the phrase "too many" as used in the foregoing sentence means more than necessary to obtain a practical near optimum design. The approximation concepts employed in the work reported here improve structural synthesis efficiency by alleviating the foregoing excesses as follows: a) the number of independent design variables may be reduced by employing design variable linking (which is in effect a special form of the reduced basis concept in design space); b) the number of constraints is reduced by considering only critical and "near" critical constraints at each stage of an iterative design procedure that tends to generate a sequence of noncritical designs of decreasing weight; and c) the number of structural analyses is reduced by employing first order Taylor series expansion to explicitly approximate the implicit dependence of the structural response on the design variables.

III. Formulation

The scope of the work presented here is limited to structural design optimization problems involving minimum weight design with respect to static stress and displacement constraints under several alternative load conditions. Attention is further restricted to that special but significant class of problems in which a) only sizing design variables (D_i) are treated; and b) element stresses are inversely proportional to the sizing design variables. The method presented for general space trusses is applicable to any structure that can be idealized using bars, shear panels, and constant strain membrane triangles.

It is well known that the minimum weight optimum design problem can be stated concisely as a mathematical programing problem as follows. Find the vector of design variables $\mathbf{D}$ such that

$$G_q(\mathbf{D}) \geqq 0 : q = 1, 2, \ldots Q \tag{1}$$

and

$$M(\mathbf{D}) \to \min \tag{2}$$

where the constraints of interest are represented by Eq. (1), $M(\mathbf{D})$ denotes the objective function and $\mathbf{D}$ is the vector of design variables (for example the cross-sectional areas A_i of truss members or the thicknesses t_i of shear panels or membrane triangles).

Reciprocal Variables

It has been found advantageous to approach the optimization problem treated here using reciprocals of the sizing type design variables D_i. Let the reciprocal design variables β_i be defined as follows:

$$\beta_i = 1/D_i \qquad i = 1, 2, \ldots I \tag{3}$$

This change of variables is motivated by the fact that for statically determinate structures idealized by elements in which the stress is inversely proportional to the sizing design variables D_i, both the stress and displacement response are strictly linear in the reciprocal variables β_i. Furthermore, using reciprocal variables generally improves the quality of the linear approximations of the stress and displacement response for moderately indeterminate structures.

The reciprocal design variables also tend to have a further advantage when it comes to dealing with minimum member size requirements. In particular it is not usually necessary to include minimum member size constraints (upper limits on the β_i) until the member size gets close to the specified minimum size, since the absence of upper limits on the β_i does not run the risk of generating negative values for any of the original variables D_i. The only apparent shortcoming of the reciprocal design variables β_i is that a linear objective function

$$M(\mathbf{D}) = \sum_{i=1}^{I} C_i D_i \tag{4}$$

becomes nonlinear in the reciprocal variables, that is

$$\tilde{M}(\boldsymbol{\beta}) = \sum_{i=1}^{I} C_i/\beta_i \tag{5}$$

However, computational experience indicates that the difficulties posed by the explicit nonlinearity exhibited in Eq. (5) are manageable through the use of move limits.

Reduced Basis

One of the most promising approximation concepts is the use of the reduced basis approach in design space. This important idea and its initial exploration was set forth in Ref. 16. The essential notion is to let the vector of design variables be expressed as a linear combination of basis vectors $\mathbf{T}_b$ as follows:

$$\boldsymbol{\beta} = \sum_{b=1}^{B} \mathbf{T}_b \alpha_b = T\boldsymbol{\alpha} \tag{6}$$

where $\mathbf{T}_b$ are I dimensional basis vectors and the α_b; $b = 1, 2, \ldots B \leqq I$ represent a reduced set of generalized design variables. In matrix form it is understood that T is an $I \times B$ matrix in which each column corresponds to a basis vector and $\boldsymbol{\alpha}$ is a vector with B elements representing the reduced set of generalized design variables. Note that Eq. (6) can be written in the following alternate form

$$\beta_i = \sum_{b=1}^{B} T_{ib} \alpha_b; \quad i = 1, 2, \ldots I \tag{7}$$

where the T_{ib} are individual elements of the T matrix.

Basis vectors can be drawn from various sources including a) design variable linking; b) optimality criteria solutions; and c) upper and lower bound solutions. It is important to recognize that design variable linking is a special form of basis reduction in which the basis vectors are chosen by the responsible engineer. When Eq. (6) is used to implement design variable linking each row of the T matrix contains only one nonzero positive element. If no design variable linking is imposed then T is simply taken as the identity matrix. While the results reported here only employ Eq. (6) as a means of imposing design variable linking the formulation and the associated computer program sets the stage for experimenting with various basis vector combinations.

Introducing the variable changes given by Eqs. (3) and (7) the optimum design problem previously stated in Eqs. (1) and (2) can be posed in terms of the generalized design variables α_b as follows:

$$H_q(\boldsymbol{\alpha}) \geqq 0 : \quad q = 1, 2, \ldots Q \tag{8}$$

and

$$W(\boldsymbol{\alpha}) \rightarrow \min \tag{9}$$

Truncated Posture Table

For any trial design $\boldsymbol{\alpha}_p$ it is a straight forward matter to execute a finite element structural analysis which yields a complete set of displacement and stress results for K load conditions. Let $Y_q(\boldsymbol{\alpha})$ represent the qth response quantity of interest, for example it could represent $\sigma_{ik}(\boldsymbol{\alpha})$ the stress in the ith truss member under the kth load condition or $u_{jk}(\boldsymbol{\alpha})$ the jth displacement degree of freedom under the kth load condition. In general limitations on the qth response quantity can be stated as follows:

$$Y_q^{(L)} \leqq Y_q(\boldsymbol{\alpha}) \leqq Y_q^{(U)} \tag{10}$$

where $Y_q^{(L)} < 0$ represents the lower limit and $Y_q^{(U)} > 0$ denotes the upper limit (e.g., the allowable compressive stress and the allowable tensile stress in a truss member). It is useful to define a quantity $R_q(\boldsymbol{\alpha})$ called the response ratio as follows:

$$R_q(\boldsymbol{\alpha}) = \begin{Bmatrix} Y_q(\boldsymbol{\alpha})/Y_q^{(U)} & \text{if} \quad Y_q(\boldsymbol{\alpha}) \geqq 0 \\ Y_q(\boldsymbol{\alpha})/Y_q^{(L)} & \text{if} \quad Y_q(\boldsymbol{\alpha}) < 0 \end{Bmatrix} \tag{11}$$

Response constraints can then be expressed in terms of response ratios as follows:

$$H_q(\boldsymbol{\alpha}) = 1 - R_q(\boldsymbol{\alpha}) \geqq 0 \tag{12}$$

The results of a finite element structural analysis for a trial design $\boldsymbol{\alpha}_p$ are used to construct a complete list of stress response ratios and a complete list of displacement response ratios. When design variable linking is employed it is efficient to retain only the most critical stress response ratio from a group of members associated with a particular generalized design variable (α_b) for each load condition. Thus all but $B \times K$ of the $I \times K$ stress response ratios are deleted. The remaining stress response ratios are placed in an ordered list of decreasing magnitude called the stress posture table. The displacement response ratios are placed in a separate ordered list of decreasing magnitude referred to as the displacement posture table. The largest response ratio in either of these two ordered lists is then identified and designated $R_1(\boldsymbol{\alpha}_p)$. Each posture table is then truncated by deleting response ratios that are less than $R_1(\boldsymbol{\alpha}_p)/R^*$, where $R^* > 1$ is an input parameter, except that the first (largest) entry is never deleted from either posture table. Unless otherwise noted, results reported herein have been obtained using $R^* = 3$. In summary then the total number of stress and displacement constraints Q is reduced to a subset of critical and potentially critical constraints $Q_R^{(p)}$, where $Q_R^{(p)}$ is usually much smaller than Q.

It is important to recognize that each time a structural analysis is executed new truncated posture tables are constructed and therefore the set of constraints guiding the design process is dynamic. As the design procedure converges the set of response ratios residing in the truncated posture tables will tend to stabilize, identifying the set of constraints that are critical and near critical for the optimum design.

IV. Design Optimization Procedure

Having reduced the number of design variables and the number of inequality constraints, attention is now focused on reducing the number of structural analyses. This is accomplished by adapting the method of inscribed hyperspheres[17] to the design optimization problem at hand. The method of inscribed hyperspheres is a sequence of linear programs approach to the optimization problem that tends to generate a sequence of noncritical designs of steadily decreasing weight. In other words, the designs obtained tend to be on a trajectory in the design space $(\boldsymbol{\alpha})$ that "funnels down the middle" of the acceptable region.

Given a trial design $\boldsymbol{\alpha}_p$ linearized approximations of the objective function $W(\boldsymbol{\alpha})$ [see Eq. (9)] and the current set of critical or potentially critical constraints $H_q(\boldsymbol{\alpha})$; $q = 1, 2, \ldots Q_R^{(p)}$ [see Eq. (8)] can be expressed as follows:

$$\tilde{W}^{(p)}(\boldsymbol{\alpha}) = W(\boldsymbol{\alpha}_p) + (\boldsymbol{\alpha} - \boldsymbol{\alpha}_p)^T \nabla W(\boldsymbol{\alpha}_p) \tag{13}$$

and

$$\tilde{H}_q^{(p)}(\boldsymbol{\alpha}) = H_q(\boldsymbol{\alpha}_p) + (\boldsymbol{\alpha} - \boldsymbol{\alpha}_p)^T \nabla H_q(\boldsymbol{\alpha}_p); \quad q = 1, 2, \ldots Q_R^{(p)} \tag{14}$$

It should be noted that the ability of first-order Taylor series approximations to predict static stress and displacement response with small error even for large design modifications has been substantiated by the investigation reported in Ref. 18. In the work reported here the quality of these linear approximations is enhanced by using reciprocal design variables.

Sensitivity Analysis

The partial derivatives which make up the elements of the gradient vector $\nabla W(\boldsymbol{\alpha})$ are easily shown to be given by

$$\frac{\partial W}{\partial \alpha_b} = -\sum_{i=1}^{I} \frac{C_i}{\beta_i^2} T_{ib} = -\sum_{i=1}^{I} C_i D_i^2 T_{ib}; \qquad b = 1, 2, \ldots B \tag{15}$$

The partial derivatives which make up the elements of the gradient vectors $\nabla H_q(\boldsymbol{\alpha})$ depend upon $\partial Y_q/\partial \alpha_b$ in view of Eqs. (11) and (12). For example if $u_{jk}(\boldsymbol{\alpha})$ (the jth displacement degree of freedom under the kth load condition) is the response quantity of interest then the pertinent partial derivatives are given by

$$\frac{\partial u_{jk}}{\partial \alpha_b} = -\sum_{i=1}^{I} \frac{1}{\beta_i^2} \frac{\partial u_{jk}}{\partial D_i} T_{ib} = -\sum_{i=1}^{I} D_i^2 \frac{\partial u_{jk}}{\partial D_i} T_{ib} \qquad b = 1, 2, \ldots B \tag{16}$$

If $\sigma_{ik}(\boldsymbol{\alpha})$ (the stress in the ith truss member under the kth load condition) is the response quantity of interest then the appropriate partial derivatives are given by

$$\frac{\partial \sigma_{ik}}{\partial \alpha_b} = \sum_{j=1}^{J} \theta_{ij} \frac{\partial u_{jk}}{\partial \alpha_b}; \qquad b = 1, 2, \ldots B \tag{17}$$

where θ_{ij} are constants that do not depend on the design variables for the class of problems considered here.

It is apparent from Eqs. (16) and (17) that the partial derivatives $\partial u_{jk}/\partial D_i$ play a central role in constructing linear approximations of the response constraints. Since a finite element displacement method of linear structural analysis is used as the basic analysis tool for predicting the behavior of trial designs the analysis of any particular design involves assembly of the system stiffness matrix $\kappa(\mathbf{D})$ and solution of the well-known equation

$$\kappa \mathbf{u}_k = \mathbf{P}_k; \qquad k = 1, 2, \ldots K \tag{18}$$

to obtain the displacement solution $\mathbf{u}_k$ for each of several load conditions $\mathbf{P}_k$. The partial derivatives $\partial u_{jk}/\partial D_i$ are easily obtained since

$$\kappa(\partial \mathbf{u}_k/\partial D_i) = -(\partial \kappa/\partial D_i)\mathbf{u}_k; \quad i = 1, 2, \ldots I; \quad k = 1, 2, \ldots K \tag{19}$$

assuming $\mathbf{P}_k$ to be independent of the design variables. The current computer program solves Eq. (18) using LU decomposition and the determination of the $\partial \mathbf{u}_k/\partial D_i$ for all I design variables is accomplished by executing I back substitutions using Eq. (19) for each load condition $k = 1, 2, \ldots K$. In order to reduce computational effort and reduce storage requirements a more efficient scheme which determines only those $\partial u_{jk}/\partial D_i$ needed should be developed. A selectivity sensitivity analysis method is being implemented and it will be reported shortly.

Sequence of LP Problems

Starting from initial trial design that is acceptable the method of inscribed hyperspheres generates a sequence of designs by finding the center of the largest hypersphere that can be inscribed within the hyperplanes representing a) the currently retained linearized behavior constraints [see Eq. (14)]; b) the current linearized approximations of the objective function [see Eq. (13)]; and c) move limits on the variables α_b, namely

$$\alpha_{pb}^{(L)} \leqq \alpha_b \leqq \alpha_{pb}^{(U)}; \qquad b = 1, 2, \ldots B \tag{20}$$

where the $\alpha_{pb}^{(L)}$ denote the lower limits and the $\alpha_{pb}^{(U)}$ denote the upper limits for the pth stage of the design optimization procedure. The basic concept of the method of inscribed hyperspheres is depicted schematically using a hypothetical two dimensional design space in Fig. 1. Let $\mathbf{S}_p$ denote the vector locating the center of the largest inscribed hypersphere relative to the current trial design $\boldsymbol{\alpha}_p$. Let l_q denote the length of a

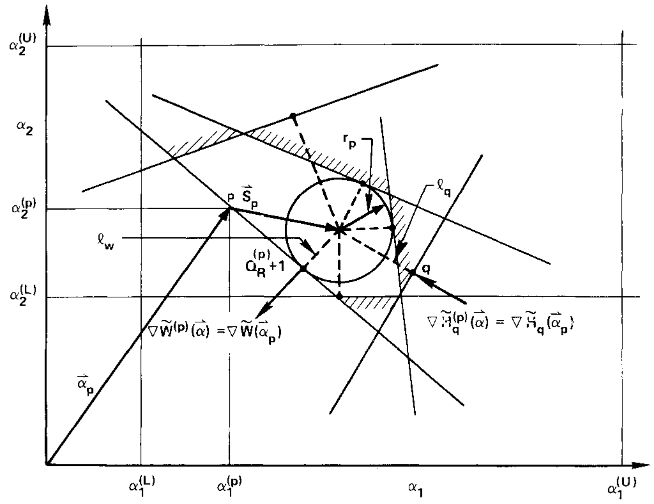


Fig. 1 Method of inscribed hyperspheres.

normal to the qth hyperplane that also passes through the center of the largest inscribed hypersphere. Then it can be shown that[17]

$$l_q = \frac{H_q(\boldsymbol{\alpha}_p) + \mathbf{S}_p{}^T \nabla H_q(\boldsymbol{\alpha}_p)}{|\nabla H_q(\boldsymbol{\alpha}_p)|}; \qquad q = 1,2,\ldots Q_R{}^{(p)} \tag{21}$$

Furthermore let l_w denote the length of a normal to the hyperplane representing the current linearized approximation of the objective function that also passes through the center of the largest inscribed hypersphere. Then it can be shown that[17]

$$l_w = -\mathbf{S}_p{}^T \nabla W(\boldsymbol{\alpha}_p)/|\nabla W(\boldsymbol{\alpha}_p)| \tag{22}$$

It is important to recognize that the expressions for l_q; $q = 1,2,\ldots Q_R{}^{(p)}$ and l_w given by Eqs. (21) and (22) are linear in $\mathbf{S}_p$. When move limits on any of the variables α_b are appropriate [see Eq. (20)] they can be added easily since the normal distance from the center of the hypersphere to the lower move limit hyperplanes $l_b{}^{(L)}$ si given by

$$l_b{}^{(L)} = \alpha_b{}^{(p)} - \alpha_{pb}{}^{(L)} + S_b{}^{(p)}; \qquad b = 1,2,\ldots B \tag{23}$$

and the normal distance from the center of the hypersphere to the upper move limit hyperplanes $l_b{}^{(U)}$ is given by

$$l_b{}^{(U)} = \alpha_{pb}{}^{(U)} - \alpha_b{}^{(p)} - S_b{}^{(p)}; \qquad b = 1,2,\ldots B \tag{24}$$

where it is understood that the $\alpha_b{}^{(p)}$ are components of the design vector $\boldsymbol{\alpha}_p$ and the $S_b{}^{(p)}$ are the components of the move vector $\mathbf{S}_p$.

The problem of locating the center of the largest inscribed hypersphere can now be cast in the following form:

Find $\mathbf{S}_p$ and r_p where r_p denotes the radius of the pth hypersphere, such that

$$l_q \geqq r_p; \qquad q = 1,2,\ldots Q_R{}^{(p)} \tag{25}$$

$$l_w \geqq r_p \tag{26}$$

$$l_b{}^{(L)} \geqq r_p; \qquad b \in B_L{}^{(p)} \tag{27}$$

$$l_b{}^{(U)} \geqq r_p; \qquad b \in B_U{}^{(p)} \tag{28}$$

and

$$r_p \to \text{Max} \tag{29}$$

where $B_L{}^{(p)}$ denotes the set of lower limits and $B_U{}^{(p)}$ denotes the set of upper limits appropriate for the pth stage of the design optimization procedure. This is clearly a linear programing problem in which the unknowns are the scalar components $S_b{}^{(p)}$ of the move vector $\mathbf{S}_p$ and the radius of the hypersphere r_p. Since the components of the move vector are unrestricted in sign the total number of nonnegative variables in each linear program is $(2B+1)$. At each stage p the number of inequality constraints included in the linear program may change. Using the truncated posture tables based on the pth structural analysis only $Q_R{}^{(p)}$ response constraints are included and one additional constraint representing the linearized objective function is included. In addition to these constraints there may be a maximum of $2B$ move limits included.

For each stage p of the design optimization procedure a linear programing problem is constructed based on Eqs. (25–29). In the current computer program these LP problems are solved using a primal-dual method referred to as the MINIT algorithm.[19] The solution of the pth LP problem gives the move vector $\mathbf{S}_p$ locating the center of the largest inscribed hypersphere and the radius of the pth hypersphere r_p. It should also be noted that the solution of each LP can be used to determine which constraint hyperplanes are in fact tangent to the hypersphere.

Move Limits

The move limit strategy employed in the current computer program is briefly described. Its objective is to exclude portions of the design space far behind the current design point while at the same time allowing sufficient freedom for the design trajectory to be formed without excessive influence from the move limits. In each stage in the sequence of linear programs a complete set of B lower limits $\alpha_{pb}{}^{(L)} \leqq \alpha_b{}^p$; $b = 1,2,\ldots B$ are included. In the first two stages these limits take the form of nonnegativity provisions placed on the generalized design variables, that is

$$\alpha_{pb}{}^{(L)} = 0 \leqq \alpha_b{}^p; \qquad b = 1,2,\ldots B; \quad p = 1,2 \tag{30}$$

For all subsequent stages the radius of the hypersphere obtained in the previous stage is used to establish the lower limits on the generalized design variables according to the scheme depicted in Fig. 2 and the lower limits are given by the following expression:

$$\alpha_{pb}{}^{(L)} = \alpha_b{}^{(p-1)} + S_b{}^{(p-1)} - r_{p-1}; \quad b = 1,2,\ldots B; \; p = 3,4\ldots \tag{31}$$

This adaptive scheme for selecting lower limits excludes regions of the design space in which the next design is not likely to reside in view of the results of previous stages. Only portions of the design space open to search at some previous stage in the design process are excluded by the lower limits employed.

The upper limit on each generalized design variable is included only if α_b shows a tendency to get close to its upper limit $\alpha_b{}^{(U)}$ within a margin. In particular the upper limit on $\alpha_b{}^{(p)}$ is included for the pth stage only if

$$[\alpha_b{}^{(U)}/\alpha_b{}^{(p-1)}] \leqq \tilde{R}^* \tag{32}$$

where $\tilde{R}^*$ is set equal to $9R^*$. Note that the subscript p has been dropped from the upper move limits, i.e., $\alpha_{pb}{}^{(U)} \to \alpha_b{}^{(U)}$ since they are only used to impose minimum size requirements that do not change from stage to stage.

In its present form the implementing computer program is limited to seeking the best design that can be represented as a positive linear combination of the given basis vectors [see Eq. (6)]. This does not represent a limitation when the basis vectors are obtained by design variable linking provided the minimum size requirements are consistent with the design variable linking. However, when the basis vectors are drawn from other sources

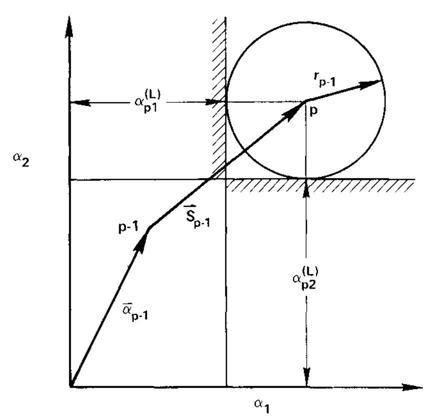


Fig. 2 Tangent plane move limits.

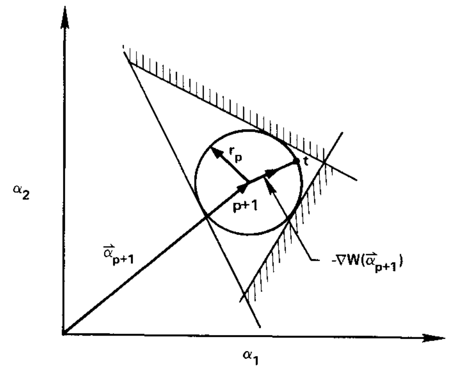


Fig. 3 First termination criterion.

limiting the search to positive linear combinations may represent a significant restriction.

Starting Point and Termination Criteria

The starting point $\boldsymbol{\alpha}_1$ for the sequence of linear programs design procedure is obtained as follows. The user supplies the program with an initial trial design $\boldsymbol{\alpha}_0$ (for examples reported herein all $\alpha_{0b} = 1$; $b = 1, 2, \ldots B$). A structural analysis of this design is executed and the corresponding posture table is generated. The initial design is then scaled

$$\boldsymbol{\alpha}_1 = C\boldsymbol{\alpha}_0 \tag{33}$$

so that the design $\boldsymbol{\alpha}_1$ has a factor of safety F_s with respect to the most critical response constraint. The starting point factor of safety F_s is an input control parameter. A value of $F_s = 1.5$ has been used to obtain the results reported herein unless otherwise specified.

Given an initial trial design $\boldsymbol{\alpha}_0$ the computer program generates a starting design $\boldsymbol{\alpha}_1$ and then proceeds to generate a sequence of designs by solving an appropriately posed LP problem for each stage p [see Eqs. (25–29)]. As in any iterative method it is necessary to establish termination criteria. There are three termination criteria employed by the current program. The first termination criterion is based on an idea illustrated in Fig. 3. At the end of the pth stage the move vector $\mathbf{S}_p$ and the hypersphere radius r_p are known and the design α_{p+1} is given by

$$\boldsymbol{\alpha}_{p+1} = \boldsymbol{\alpha}_p + \mathbf{S}_p \tag{34}$$

Referring to Fig. 3 let t denote the point where the negative gradient vector $-\nabla W(\boldsymbol{\alpha}_{p+1})$ emanating from point $p+1$ intersects the hypersphere of radius r_p. The first termination criterion is said to be satisfied when the weight at design $\boldsymbol{\alpha}_{p+1}$ minus the first-order Taylor series estimate of the weight at point t divided by the starting point weight $W(\boldsymbol{\alpha}_1)$ is less than an input tolerance ε_1, that is when

$$\frac{W(\boldsymbol{\alpha}_{p+1}) - \tilde{W}^{(p+1)}(\boldsymbol{\alpha}_t)}{W(\boldsymbol{\alpha}_1)} = \frac{r_p |\nabla W(\boldsymbol{\alpha}_{p+1})|}{W(\boldsymbol{\alpha}_1)} \leqq \varepsilon_1 \tag{35}$$

for two consecutive stages. A representative value of ε_1 used to obtain the results reported herein is $\varepsilon_1 = 0.01$.

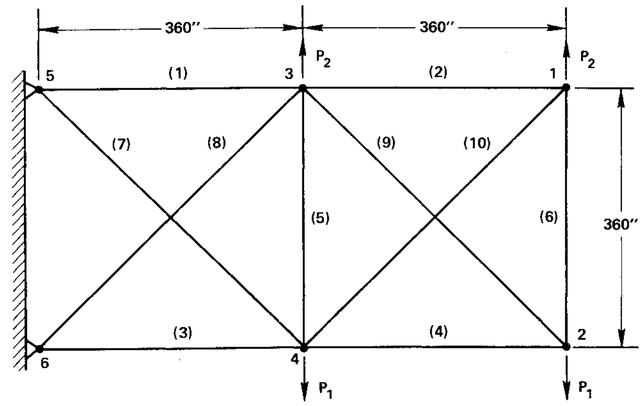
At the user's option a termination phase may be called for. If this option is elected, the program seeks to refine the results after satisfaction of the first termination criterion in the following manner. The LP problem given by Eqs. (25–29) is altered by deleting the radius of hypersphere from consideration and taking the current linearized approximation of the weight as the objective function to be minimized. Note that the constraint of Eq. (26) would thus be redundant and it is therefore omitted. When this option is elected at least two LP stages are executed. The second termination criterion is said to be satisfied if

$$\frac{|W(\boldsymbol{\alpha}_p) - W(\boldsymbol{\alpha}_{p+1})|}{W(\boldsymbol{\alpha}_{p+1})} \leqq \varepsilon_2 \tag{36}$$

A representative range of values for ε_2 used to obtain the results reported herein is $0.002 \leqq \varepsilon_2 \leqq 0.01$. The third termination criterion is a simple user supplied limit on the number of LP stages. This overriding termination criterion is simply stated as

$$p \leqq p_{\max} \tag{37}$$

where $p_{\max}$ is the maximum number of LP stages to be permitted (typically taken as $p_{\max} = 30$ for the results reported herein).



MATERIAL: ALUMINUM, E = 10^7 psi, ρ = .1 pci
STRESS LIMITS: ± 25000. psi (ALL MEMBERS)
LOWER LIMITS: .1 in^2 (ALL MEMBERS)
UPPER LIMITS: NONE

LOADING CASE 1: SINGLE LOAD P_1 = 100.K, P_2 = 0.
LOADING CASE 2: SINGLE LOAD P_1 = 150.K, P_2 = 50.K

Fig. 4 Ten bar truss.

V. Numerical Examples

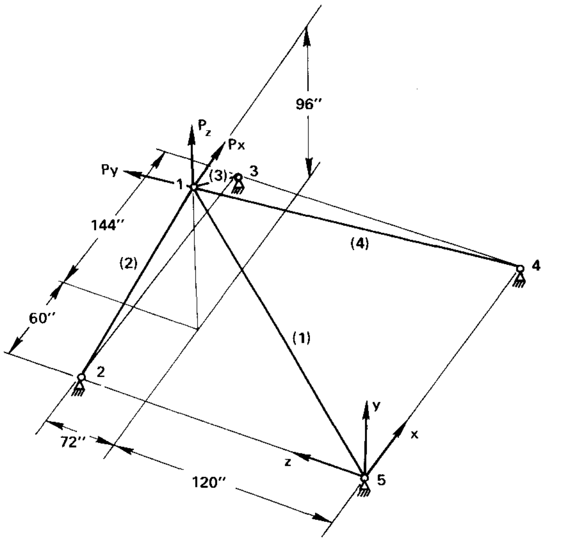
The effectiveness of the approximation concepts employed is illustrated by presenting results for several previously studied two- and three-dimensional truss examples. The results reported here have been obtained using a Fortran H computer program that implements the design procedure previously described. The computations were carried out on an IBM 360/91 computer and required 302K bytes of core storage.

Example 1

The first example problem is a familiar ten member planar truss (see Fig. 4) for which results have been previously

Table 1 Summary of results for example 1—10 bars

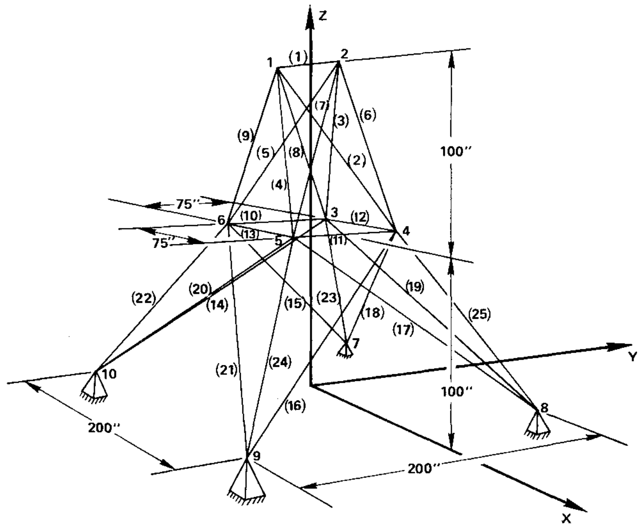
Case	Minimum weight (lbs)	No. of analyses	CPU runtime (sec)	Member areas for optimum design (in.^{2})									
				1	2	3	4	5	6	7	8	9	10
1a	1593.2	20	0.9	7.938	0.1	8.062	3.938	0.1	0.1	5.745	5.569	5.569	0.1
1b	5089.0	23	1.0	33.432	0.1	24.260	14.260	0.1	0.1	8.338	20.740	19.690	0.1
2a	1664.5	20	0.9	5.948	0.1	10.052	3.948	0.1	2.052	8.559	2.754	5.583	0.1
2b	4691.8	22	1.0	24.290	0.1	23.346	13.654	0.1	1.970	12.670	12.544	21.971	0.1



MATERIAL: ALUMINUM, E = 10^7 psi, ρ = .1 pci
STRESS LIMITS: ± 25000, psi ON ALL MEMBERS
LOWER LIMITS: NONE
UPPER LIMITS: NONE

LOADING CASE 1: SINGLE LOAD P_x = 10.K, P_y = 20.K, P_z = -60.K
LOADING CASE 2: SINGLE LOAD P_x = 40.K, P_y = 100.K, P_z = -30.K

Fig. 5 Four bar truss.



MATERIAL: ALUMINUM, E = 10^7 psi, ρ = .1 pci
STRESS LIMITS: TENSION +40000, psi (SEE TABLE FOR COMPRESSION)
LOWER LIMITS: .01 in^2
UPPER LIMITS: NONE
DISPLACEMENT LIMITS: .35 in. ON ALL NODES AND IN ALL DIRECTIONS

Fig. 6 Twenty-five bar truss.

reported.[10,11] The material properties, stress limits, minimum member sizes, and load condition data for this example are given in Fig. 4 and results are given for four distinct cases in Table 1. Cases 1a and 2a involve stress and minimum member size limits only while Cases 1b and 2b include vertical displacements limits of ± 2.0 in at all points in addition to the stress and minimum member size constraints. No design variable linking has been employed in this example. The minimum weights obtained are essentially the same as those previously reported except in Case 2b where the result obtained here is 204 lb lower than that reported in Ref. 10. It is interesting to note that the number of analyses needed to obtain a result is almost invariant over these four cases ranging from 20 to 23. This is consistent with the expectation that the convergence characteristics of the design procedure presented here would tend to be independent of the type (stress or displacement) of response constraints active at the optimum.

Table 2 Summary of results for example 2—4 bars

Case	Minimum weight (lbs)	No. of analyses	CPU runtime (sec)	Member areas for optimum design (in.^{2}) 1	2	3	4
1a	65.76	13	0.3	0.858	1.406	1.745	0.000
1b	117.89	16	0.4	0.000	3.765	0.769	2.514
2a	115.26	14	0.3	2.663	2.298	2.159	0.000
2b	128.53	14	0.3	3.210	2.614	2.159	0.000

Example 2

The second example problem is a four bar space truss (see Fig. 5) for which results have been previously reported in Refs. 10 and 11. The material properties, stress limits and load condition data for this example are given in Fig. 5 and results for four distinct cases are given in Table 2. Cases 1a and 2a involve stress limits only while Case 1b includes stress and displacements limits (z direction limits at joint 1 are ± 0.3 in.) and Case 2b has stress and displacement limits as follows (at joint 1; x direction limits ± 0.3 in., y direction limits ± 0.5 in., z direction limits ± 0.4 in.). No design variable linking has been employed in this example. The minimum weights obtained are essentially the same as those previously reported in Ref. 10 except in Case 1b where Ref. 10 reports a weight of 126.43 lb after 37 analyses as compared with 117.89 lb after 16 analyses by the present procedure. It is again noted that the number of analyses needed to obtain a result is almost invariant over these four cases ranging from 13 to 16. This further supports the contention that the effort required for convergence of the procedure presented

Table 3 Summary of results for example 3—25 bars

Minimum weight (lbs)	No. of analyses	CPU runtime (sec)	Member design data at optimum: Members	1	2, 3, 4, 5	6, 7, 8, 9	10, 11	12, 13	14, 15, 16, 17	18, 19, 20, 21	22, 23, 24, 25
545.23	16	2.5	Member areas (in.^{2})	0.01	1.964	3.033	0.01	0.01	0.670	1.680	2.670
			Allowable stress (psi)	−35092	−11590	−17305	−35092	−35092	−6759	−6959	−11082

Table 4 Load conditions for example 3—25 bars

Load condition	Node	Direction X	Direction Y	Direction Z
1	1	1000	10000	−5000
	2	0	10000	−5000
	3	500	0	0
	6	500	0	0
2	1	0	20000	−5000
	2	0	−20000	−5000

Table 5 Load conditions for example 4—72 bars

Load condition	Node	Direction X	Direction Y	Direction Z
1	1	5000	5000	−5000
2	1	0	0	−5000
	2	0	0	−5000
	3	0	0	−5000
	4	0	0	−5000

here tends to be independent of the constraint mix active at the optimum.

Example 3

The third example problem is a 25 member space truss (see Fig. 6) for which results have been previously reported in Refs. 10 and 11. The material properties, tension stress limits, displacement limits, and minimum member sizes for this example are given in Fig. 6. Note that the allowable compressive stresses are listed in Table 3 and they correspond to those given in Table 2 of Ref. 11. Design variable linking is used to impose symmetry with respect to both the y-z and the x-z planes and the structure is subjected to two distinct load conditions as given in Table 4. The number of independent design variables is eight. Results for a single case in which displacement limits of ± 0.35 in. are imposed on all joints in all directions are summarized in Table 3. As indicated a minimum weight design with $W = 545.2$ lb is obtained after 16 analyses with a CPU time of 2.5 sec. The approximate distribution of effort (based on CPU time monitoring) between the main portions of the design procedure is as follows: 1) structural analyses 25%; 2) sequence of linear programs 25%; 3) generation of truncated posture tables 5%; and 4) sensitivity analyses 45%. This breakdown suggests that further gains in efficiency could be achieved by the implementation of a selective sensitivity analysis scheme. Finally, it is noted that the critical constraints at the optimum design in this case are found to be horizontal deflection at joints 1 and 2 in load conditions 1 and 2 as well as the compressive stress in member 20 in load condition 2.

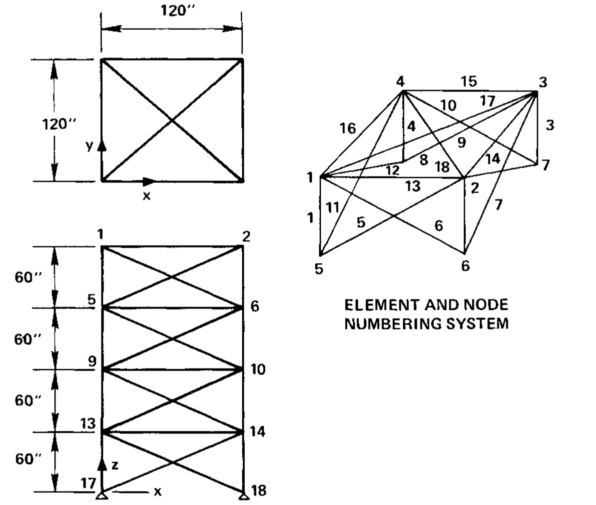


Fig. 7 Seventy-two member truss.

Example 4

The fourth example problem is a 72 member space truss for which results have been previously reported in Refs. 10 and 11. Figure 7 shows the geometry of the structure and the node as well as member numbering system is illustrated in detail for the uppermost tier. The problem as posed in Ref. 10 involves five loading conditions. The symmetry of the structure and the loading conditions are such that the number of load conditions can be reduced to the two given in Table 5 and the number of independent design variables can be reduced to 16 using design variables linking. The material properties, stress allowables, and minimum member sizes are given on Fig. 7. The displacements of nodes 1–4 are limited to ± 0.25 in. in the x and y directions. Results for this single case are summarized in Table 6. As indicated a minimum weight design with $W = 388.6$ lb is obtained after 22 analyses with a CPU time of 15.8 sec. The approximate distribution of effort (based on CPU time monitoring) between the main portions of the design procedure is as

Table 6 Summary of results for example 4—72 bars

Minimum weight (lbs)	No. of analyses	CPU runtime (sec)	Member design data at optimum								
			Members	1, 2, 3, 4	5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18	19, 20, 21, 22	23, 24, 25, 26, 27, 28, 29, 30	31, 32, 33, 34	35, 36
			Areas (in.^{2})	0.158	0.594	0.341	0.608	0.264	0.548	0.1	0.151
388.6	22	15.8	Members	37, 38, 39, 40	41, 42, 43, 44, 45, 46, 47, 48	49, 50, 51, 52	53, 54	55, 56, 57, 58	59, 60, 61, 62, 63, 64, 65, 66	67, 68, 69, 70	71, 72
			Areas (in.^{2})	1.107	0.579	0.1	0.1	2.078	0.503	0.1	0.1

follows: 1) structural analyses 17%; 2) sequence of linear programs 17%; 3) generation of truncated posture tables 2%; and 4) sensitivity analyses 64%. This breakdown reinforces the notion that further gains in efficiency could be achieved by improving the sensitivity analysis. Finally, it is noted that the critical constraints at the optimum design in this case are found to be the deflection at joint 1 in the x and y direction under load condition 1 as well as the compressive stress in member 4 under load condition 2.

VI. Conclusions

Through the use of some approximation concepts a mathematical programing type optimization procedure applicable to a significant class of static structural design problems has been generated. The number of analyses required to converge the design optimization procedure is reasonable (ranging from 13 to 22) and the computer CPU run times are practical (ranging from 0.3 sec to 15.8 sec). The example results indicate that the number of analyses required to obtain convergence tends to be essentially independent of the mix of active constraints governing the optimum design (see Tables 1 and 2). Perhaps the most remarkable characteristic of the method presented is that the number of structural analyses needed to find an optimum design appears to be relatively insensitive to the number of design variables.

In all cases executed to date for which previously reported results are available designs of essentially equal or lower weight have been found and the number of structural analyses required have been equal to or less than 22. An important practical characteristic of the procedure reported is that it tends to generate a sequence of noncritical designs of decreasing weight that approximates the free trajectory through the acceptable region of the design space.

It has been shown how some approximation concepts can be used to generate an efficient structural synthesis capability. The reduced basis concept, design variable linking in particular, provides a means of reducing the number of design variables while simultaneously imposing practical constraints on the design variables. The notion of a truncated posture table makes it possible to guide the design process at each stage taking into account only critical and potentially critical response ratios. However, it should be emphasized that, properly implemented, this technique leads to a final design that is acceptable with respect to all constraints considered in the original problem statement. Finally, the number of structural analyses is reduced by using Taylor series expansions to explicitly approximate the implicit dependence of response quantities on the design variables. In this connection it was found advantageous to introduce reciprocal variables because this impoves the quality of the linear approximations for displacement and stress response. Using the method of inscribed hyperspheres a sequence of LP problems is constructed and the solutions generated form a sequence of designs that "funnel down the middle" of the acceptable region in design space (i.e., approximate the free trajectory). It is concluded that the results reported support the contention that the innovative use of approximation concepts in structural synthesis can lead to a new generation of philosophically attractive, efficient, and practical design optimization capabilities.

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